Integer programming models and exact methods for the two-dimensional two-staged knapsack problem

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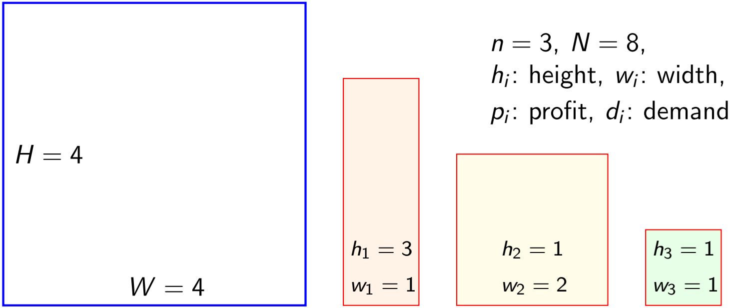
Introduction

Two-dimensional Two-staged Knapsack Problem

$n = 3,\ N = 8,$
h_i: height, w_i: width,
p_i: profit, d_i: demand



Only one plate is given

$d_1 = 3$, $p_1 = 7$

$d_2 = 2$, $p_2 = 10$

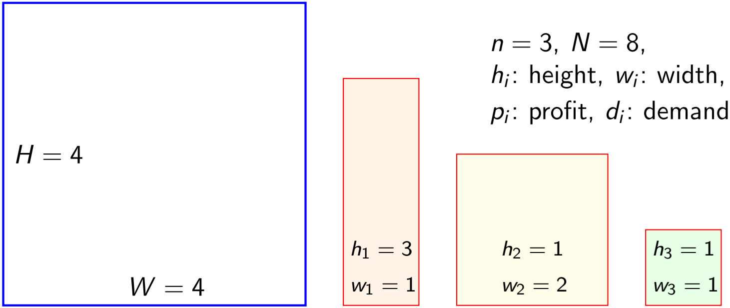
$d_3 = 3$, $p_3 = 2$

A General Two-dimensional Knapsack Problem

What is the maximum profit obtained by cutting small items from the large plate?

Two-dimensional Two-staged Knapsack Problem

$n = 3,\ N = 8,$
h_i: height, w_i: width,
p_i: profit, d_i: demand



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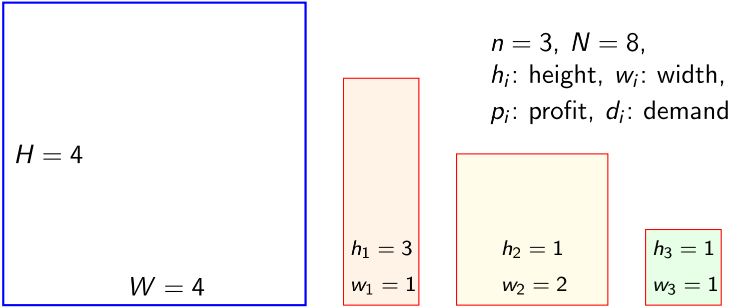
A **Two-dimensional Two-staged** Knapsack Problem

What is the maximum profit obtained by cutting small items from the large plate using **two-stage guillotine cuts**?

Examples of Two-staged Cutting

$n = 3, \; N = 8,$
h_i: height, w_i: width,
p_i: profit, d_i: demand



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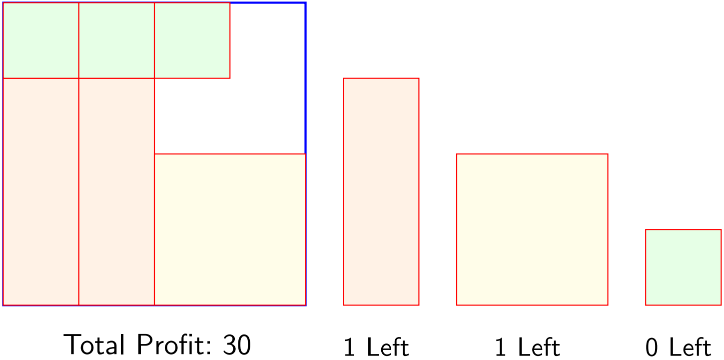

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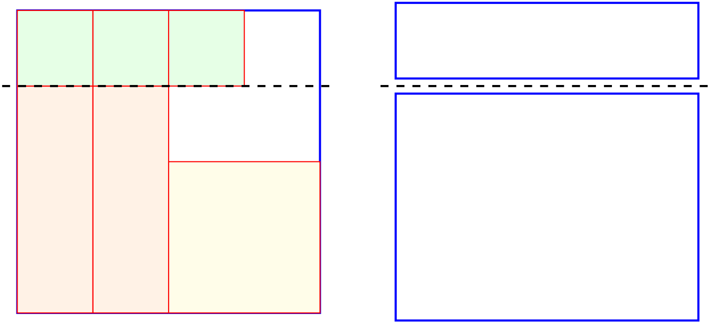
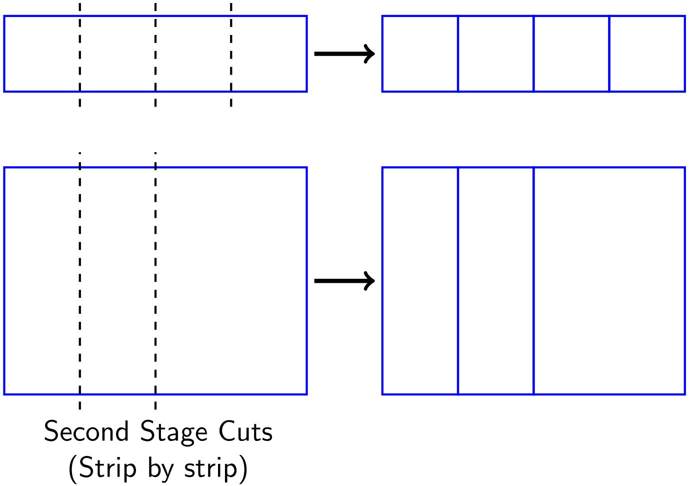


First Stage Cuts

Strips (Levels)

Examples of Two-staged Cutting



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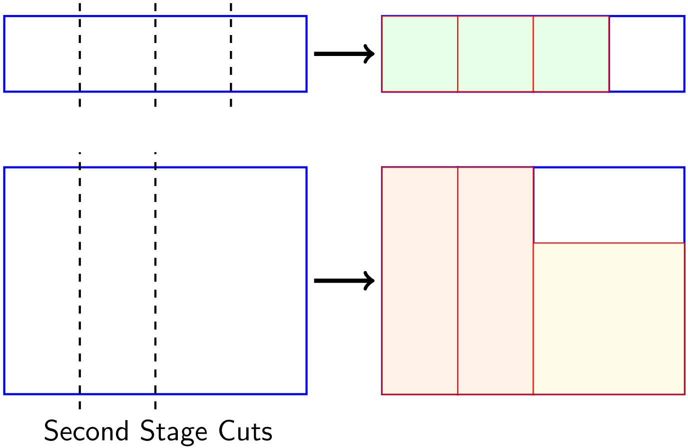

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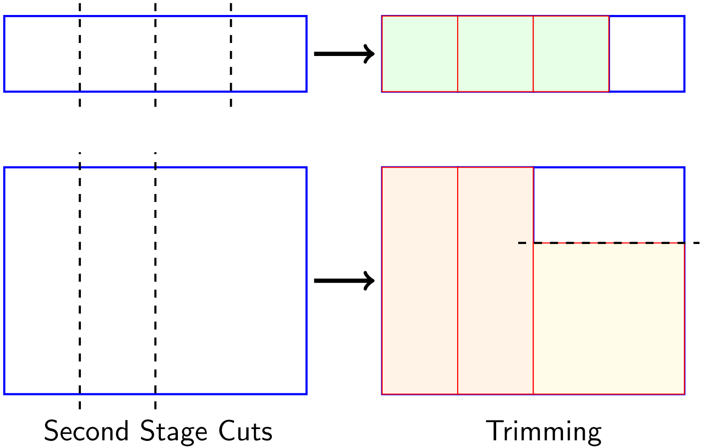
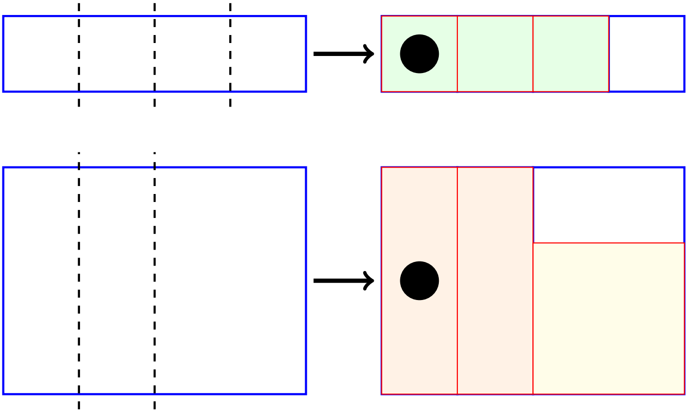


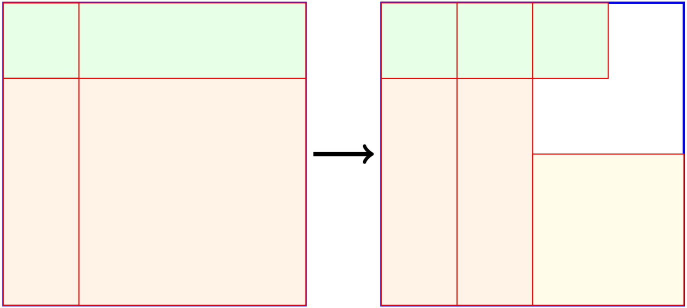
Examples of Two-staged Cutting



Black: Strip Defining Items

Examples of Two-staged Cutting

1. Set strip-defining items first.
2. Locate the rest of the items (with lower heights) into each strip.



Second Stage Cuts → Trimming

Two-dimensional Two-staged Knapsack Problem

- Industrial guillotine cutters: **efficiency** and **accuracy**.
  - One of the restrictions is two-staged guillotine cutting.

Two-dimensional Two-staged Knapsack Problem

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- Close relationship with the two-dimensional two-staged guillotine cutting stock problem
  - Sharing the same two-staged guillotine cutting constraint
  - Knapsack problem: a **slave problem**

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    - One of the restrictions is two-staged guillotine cutting.
- Close relationship with the two-dimensional two-staged guillotine cutting stock problem
    - Sharing the same two-staged guillotine cutting constraint
    - Knapsack problem: a **slave problem**
- NP-hard in a strong sense
    - Reduction from the 3-PARTITION problem
    - Unanswered research issues exist yet.

Literature Review

1. Integer programming models:
   - Gilmore and Gomory (1965) [12]: Strip Packing Model
   - Lodi and Monaci (2004) [18]: Level Packing Model

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   - Belov and Scheithauer (2006) [2]: Branch-and-cut-and-price Algorithm

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   - Belov and Scheithauer (2006) [2]: Branch-and-cut-and-price Algorithm
3. Heuristics:
   - Hifi and M'Hallah (2006) [14]: Strip Generation Algorithm
   - Alvarez-Valdes *et al.* (2007) [1]: Path Relinking Methods.

Contributions

1 Introduced new formulations for the problem based on concepts proposed for the two-dimensional two-staged cutting stock problem

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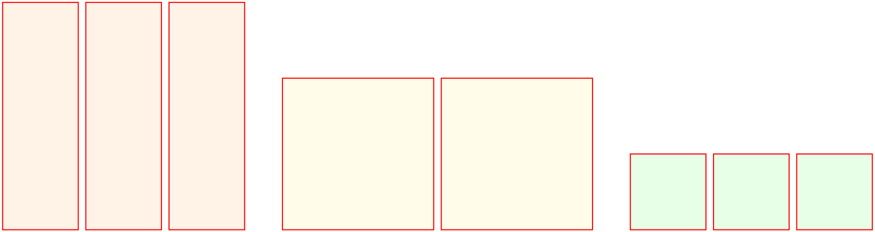
1. Introduced new formulations for the problem based on concepts proposed for the two-dimensional two-staged cutting stock problem
2. Proved the existence of polynomial-size formulation
3. Established a hierarchy of the LP-relaxation values
4. Developed some exact methods and tested them computationally.

Integer Programming Models

Existing Models: (1) A Level Packing Model (LM)

- Lodi and Monaci (2004) [18]

1. Determine which items will be used as strip-defining items.
2. Then, pack the rest of the items.



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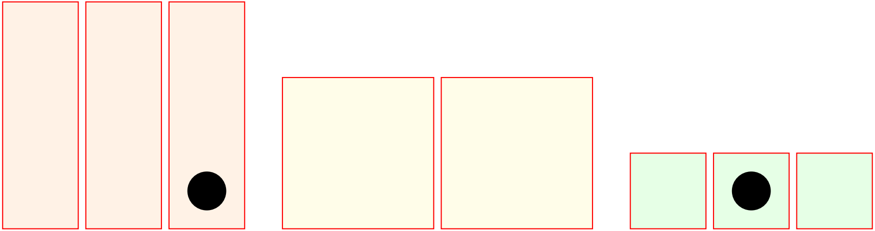


Existing Models: (1) A Level Packing Model (LM)

$$
\begin{aligned}
\text{LM}: \quad \text{maximize} \quad & \sum_{j=1}^{N} p_{\beta_k} \sum_{k=1}^{j} x_{jk} \\
\text{subject to} \quad & \sum_{k=1}^{j} x_{jk} \leq 1, \quad \forall j \in \{1, \ldots, N\} \\
& \sum_{j=k+1}^{N} w_{\beta_j} x_{jk} \leq (W - w_{\beta_k}) x_{kk}, \quad \forall k \in \{1, \ldots, N\} \\
& \sum_{k=1}^{N} h_{\beta_k} x_{kk} \leq H, \\
& x_{jk} \in \{0, 1\}, \quad \forall k \in \{1, \ldots, N\}, \quad \forall j \in \{k, \ldots, N\}.
\end{aligned}
$$

- The size of the formulation depends on the total numbers of items N, not the number of item types n. (Pseudo-polynomial-size model)

Modification of a Level Packing Model

We add the following set of valid inequalities to LM:

$$x_{jk} \leq x_{kk}, \quad \forall k \in \{1, \dots, N\}, \quad \forall j \in \{k+1, \dots, N\}.$$

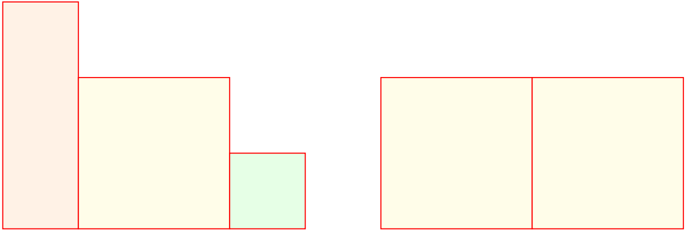
(In a strip, each item should not be used more than the strip-defining item.)

- Improve the quality of the LP-relaxation value
- Ease analyzing the relationship between other models.

Existing Models: (2) A Strip Packing Model (PM)

- Gilmore and Gomory (1965) [12]

1. Width patterns: the combination of items whose total width is below W.
2. One width pattern $\rightarrow$ One strip
3. Pack these width patterns with their total height below H.



Width Pattern $(1, 1, 1)$, Height: 3

Width Pattern $(0, 2, 0)$, Height: 2

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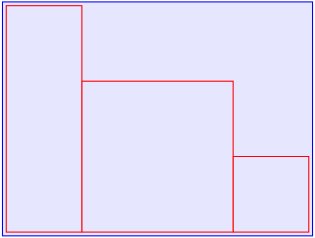

Width Pattern $(1, 1, 1)$,
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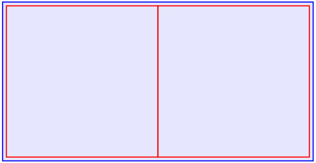


Width Pattern $(0, 2, 0)$,
Height: 2

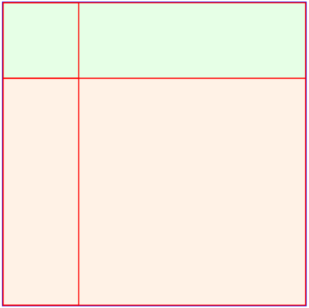
Existing Models: (2) A Strip Packing Model (PM)

$$
\begin{aligned}
\text{PM}: \quad & \text{maximize} && \sum_{q \in P_W(d)} \sum_{i \in I_n} p_i a_{qi} x_q \\
& \text{subject to} && \sum_{q \in P_W(d)} a_{qi} x_q \le d_i, \quad \forall i \in I_n, \\
& && \sum_{q \in P_W(d)} h_{t(q)} x_q \le H, \\
& && x_q \in \mathbf{Z}_+, \quad \forall q \in P_W(d).
\end{aligned}
$$

- Exponentially many width patterns exist. (Exponential-size model)

Proposed Models: (3) A Staged-pattern Model (SM)

1. Mrad *et al.* (2013) [22]
   - Propose the concept of height patterns at the two-staged two-dimensional cutting stock problem
2. Height patterns: the combination of strip-defining items whose total height is below H.
3. Choose adequate width patterns to complete the cutting.



Height Pattern $(1, 0, 1)$

Proposed Models: (3) A Staged-pattern Model (SM)

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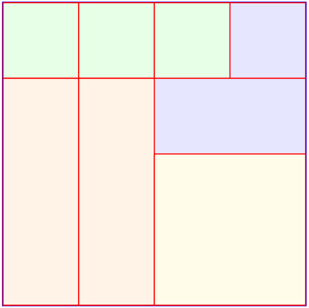


Width Pattern $(0, 0, 3), (2, 1, 0)$

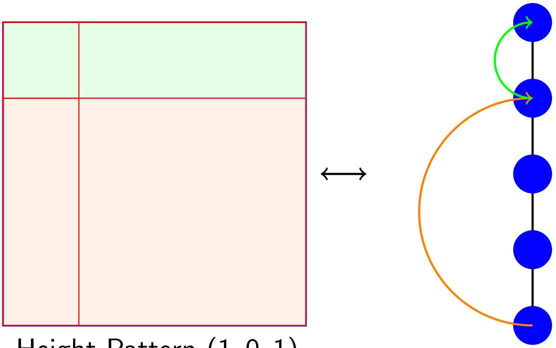
New Models: (3) A Staged-pattern Model (SM)

$$
\begin{aligned}
\text{SM}: \quad & \text{maximize} && \sum_{q \in P_W(d)} \sum_{i \in I_n} p_i a_{qi} x_q \\
& \text{subject to} && \sum_{q \in P_W(d)} a_{qi} x_q \le d_i, && \forall i \in I_n, \\
& && \sum_{r \in P_H(d)} b_{ri} y_r \ge \sum_{q \in P_W(d), t(q)=i} x_q, && \forall i \in I_n, \\
& && \sum_{r \in P_H(d)} y_r \le 1, \\
& && x_q \in \mathbf{Z}_+, \quad \forall q \in P_W(d), \\
& && y_r \in \mathbf{Z}_+, \quad \forall r \in P_H(d).
\end{aligned}
$$

- Exponentially many width patterns and height patterns exist. (Exponential-size model)

New Models: (4) An Arc-flow Model (AF)

1. Macedo *et al.* [20]:
   - Extend the concept of arc-flow model to the two-staged two-dimensional cutting stock problem
2. Represent patterns as the flow of the certain graph



Height Pattern $(1, 0, 1)$ $\longleftrightarrow$ Height Pattern Graph

New Models: (4) An Arc-flow Model (AF)

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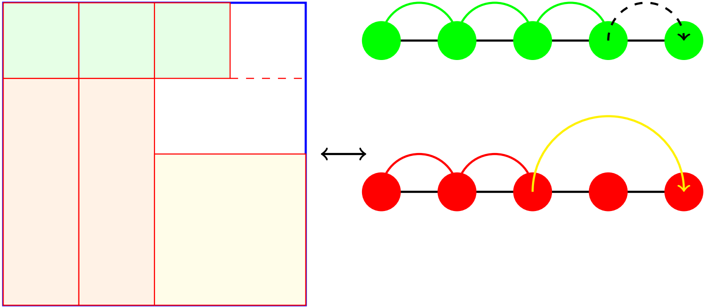

Width Pattern (0, 0, 3)
Width Pattern (2, 1, 0)

Width Pattern Graphs

New Models: (4) An Arc-flow Model (AF)

$$
\begin{aligned}
\text{AF}: \quad \text{maximize} \quad & \sum_{j \in I_n} \sum_{(a,b,i) \in A^j} \pi^j_{(a,b,i)} x^j_{(a,b,i)} \\
\text{subject to} \quad & \sum_{(a,b,i) \in A^0} x^0_{(a,b,i)} - \sum_{(b,c,k) \in A^0} x^0_{(b,c,k)} = \begin{cases} -1 & \text{if } b = 0 \\ 0 & \text{if } b = 1, \dots, H-1 \\ 1 & \text{if } b = H \end{cases}, \\
& \sum_{(c,c+h_j,k) \in A^0} x^0_{(c,c+h_j,k)} - z^j = 0, \quad \forall j \in I_n, \\
& \sum_{(d,e,i) \in A^j} x^j_{(d,e,i)} - \sum_{(e,f,k) \in A^j} x^j_{(e,f,k)} \\
& = \begin{cases} -z^j & \text{if } e = 0 \\ 0 & \text{if } e = 1, \dots, W-1 \\ z^j & \text{if } e = W \end{cases}, \quad \forall j \in I_n, \\
& \sum_{j \in I_n} \sum_{(f,f+w_i,i) \in A^j} x^j_{(f,f+w_i,i)} \le d_i, \quad \forall i \in I_n, \\
& \text{All variables are nonnegative integers.}
\end{aligned}
$$

- The size of the formulation depends on H and W. (Pseudo-polynomial-size model)

Theoretical Analysis

Existence of a Polynomial-size Formulation

Eisenbrand and Shmonin [8] provides the upper bound on numbers of nonzero components in the optimal solution in integer linear programming problems.

Lemma 1

Let p_{max} and d_{max} indicate the maximum value of components in given p and d, respectively. Then, there exists an optimal solution with at most $M = \lceil log_2(n) + log_2(p_{max}) + (n+1)log_2(d_{max}) + log_2(H) \rceil$ different types of width patterns.

- Split the integer variables into binary variables.
- Transform the multiplication of binary variables into linear constraints with a new binary variable.

Existence of a Polynomial-size Formulation: POLY

$$
\begin{aligned}
\text{maximize} \quad & \sum_{i=1}^{n}\sum_{m=1}^{M}\sum_{k=1}^{\hat{D}}\sum_{l=1}^{\hat{D}} 2^{k+l-2} p_i s_{ikml} \\
\text{subject to} \quad & \sum_{m=1}^{M}\sum_{k=1}^{\hat{D}}\sum_{l=1}^{\hat{D}} 2^{k+l-2} s_{ikml} \le d_i, \quad \forall i \\
& \sum_{i=1}^{n}\sum_{k=1}^{\hat{D}} 2^{k-1} w_i \bar{q}_{ik}^{m} \le W, \quad \forall m, \\
& \sum_{m=1}^{M}\sum_{l=1}^{\hat{D}}\sum_{t=1}^{\hat{H}} 2^{l+t-2} r_{mlt} \le H, \\
& \sum_{k=1}^{\hat{D}} 2^{k-1} \bar{q}_{ik}^{m} \le d_i z_i^{m}, \quad \forall i, m,
\end{aligned}
$$

$$
\begin{aligned}
& \bar{q}_{ik}^{m} \le z_i^{m}, \quad \forall i, k, m, \\
& \sum_{t=1}^{\hat{H}} 2^{t-1} \bar{H}_{mt} \ge h_i z_i^{m}, \quad \forall i, m, \\
& s_{ikml} \ge \bar{q}_{ik}^{m} + x_{ml} - 1, \quad \forall i, k, m, l, \\
& s_{ikml} \le \bar{q}_{ik}^{m}, \quad \forall i, k, m, l, \\
& s_{ikml} \le x_{ml}, \quad \forall i, k, m, l, \\
& r_{mlt} \ge x_{ml} + \bar{H}_{mt} - 1, \quad \forall m, l, t, \\
& r_{mlt} \le x_{ml}, \quad \forall m, l, t, \\
& r_{mlt} \le \bar{H}_{mt}, \quad \forall m, l, t, \\
& \text{all variables } s, \bar{q}, \bar{H}, r, z \text{ are binary.}
\end{aligned}
$$

- Many logical constraints $\rightarrow$ weak upper bound
- The first polynomial-sized model
  - $\mathcal{O}(n \log_2(d_{\max})^2 (n \log_2(d_{\max}) + \log_2(p_{\max}) + \log_2(H)))$

Upper bounds provided by the LP-relaxations

Optimal objective value: z^*
Optimal objective value of the LP relaxation of model "M": z^{M}

Theorem 1

$z^* \leq z^{\mathrm{SM}} \leq z^{\mathrm{PM}} \leq z^{\mathrm{LM}}$

Theorem 2

$z^* \leq z^{\mathrm{SM}} \leq z^{\mathrm{AF}}$

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LM Explicitly choose strip-defining items and then construct strips.

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- **LM** Explicitly choose strip-defining items and then construct strips.
- **PM** Packing predefined strips.
- **SM** Packing strips with a predefined height pattern.

Upper bounds provided by the LP-relaxations

Optimal objective value: z^*
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Theorem 1

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- **LM** Explicitly choose strip-defining items and then construct strips.
- **PM** Packing predefined strips.
- **SM** Packing strips with a predefined height pattern.
- **AF** Use graphs to indicate which patterns are used.

Upper bounds provided by the LP-relaxations

Because of the knapsack structure, the following theorem holds:

Theorem 3

$z^{\mathrm{LM}} \le 2z^{\mathrm{PM}} \le 4z^{\mathrm{SM}}$.

Proof.

$z^{\mathrm{LM}} \le 2z^{\mathrm{PM}}$: Compensation for a partly used item in each strip

$z^{\mathrm{PM}} \le 2z^{\mathrm{SM}}$: Compensation for a partly used strip in a plate □

Also, there exists a tight example of the above inequality.

Upper bounds provided by the LP-relaxations

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Also, there exists a tight example of the above inequality.

$$\rightarrow z^* \leq z^{\mathrm{SM}} \leq z^{\mathrm{PM}} \leq z^{\mathrm{LM}} \leq 2z^{\mathrm{PM}} \leq 4z^{\mathrm{SM}}$$

Computational Analysis

Exact Methods

- LM: Branch-and-cut (Delayed constraint generation)

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- PM: Branch-and-price
- SM: Branch-and-price

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- LM: Branch-and-cut (Delayed constraint generation)
- PM: Branch-and-price
- SM: Branch-and-price
- AF: Branch-and-bound
- POLY: Branch-and-bound

Delayed Constraint Generation of LM

We add the following set of valid inequalities to the original form of LM:

$$x_{jk} \le x_{kk}, \quad \forall k \in \{1, \ldots, N\}, \quad \forall j \in \{k+1, \ldots, N\}.$$

- The number of added inequalities: $\mathcal{O}(N^2)$

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2 At each node, solve the LP-relaxation and check whether violated inequalities exist.

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- The number of added inequalities: $\mathcal{O}(N^2)$

1. Discard the inequalities at the beginning.
2. At each node, solve the LP-relaxation and check whether violated inequalities exist.
3. Add violated inequalities to the descendants of the node.

Branch-and-price Algorithms of PM and SM

- The number of pattern variables: $\mathcal{O}(2^n)$.

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1 Start with the basic pattern variables.

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1. Start with the basic pattern variables.
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Branch-and-price Algorithms of PM and SM

- The number of pattern variables: $\mathcal{O}(2^n)$.

1. Start with the basic pattern variables.
2. For each node, solve the LP-relaxation and check whether more pattern variables are needed.
3. If needed, generate new pattern variables and repeat the procedure.

Computational Experiments

- Solvers offered by Xpress 8.9 [9]
- Intel(R) Core(TM) i7-4770 CPU @ 3.10GHz and 16GB of RAM
- Time Limit: 600 s.
- Benchmark instances proposed by Hifi and Roucairol (2001) [15]:
    - Small: 16 instances ($40 \times 40 - 130 \times 130$)
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- Intel(R) Core(TM) i7-4770 CPU @ 3.10GHz and 16GB of RAM
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Results: Small Instances

- Except for POLY, all models solved all instances to optimality

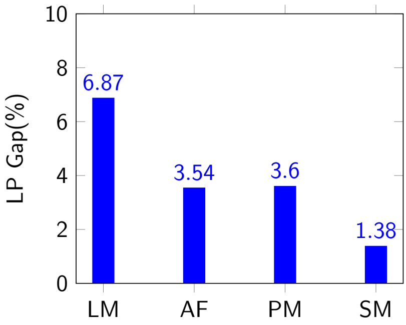

Figure: Average LP gaps.

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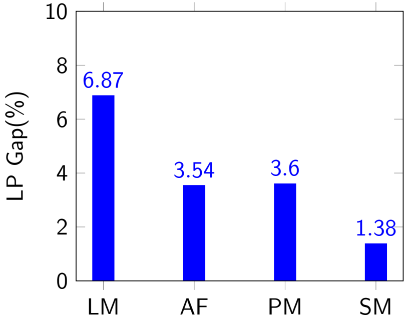

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- z^{AF} and z^{PM}: incomparable

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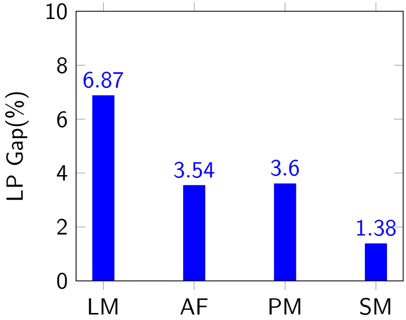

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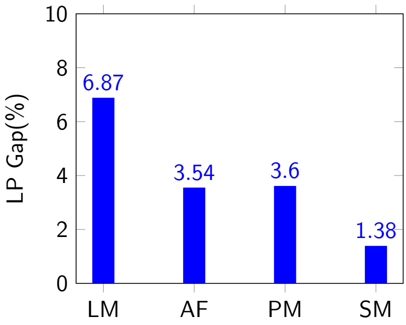

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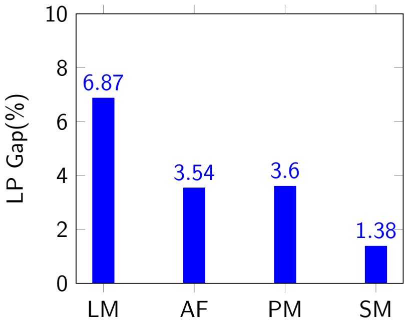

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- LP Gap of POLY: 517.5 (%)

Results: Large Instances

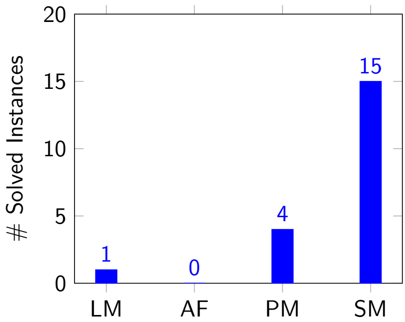

Figure: The number of solved instances.

- AF, LM: vulnerable to n and N

Results: Large Instances

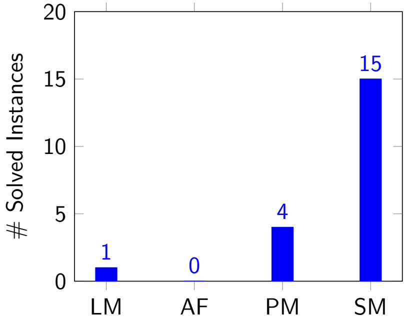

Figure: The number of solved instances.

- AF, LM: vulnerable to n and N
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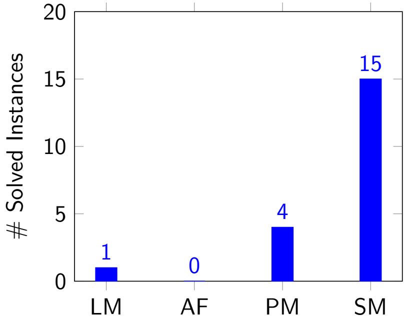
Results: Large Instances



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- Effectiveness of height patterns

Results: Large Instances

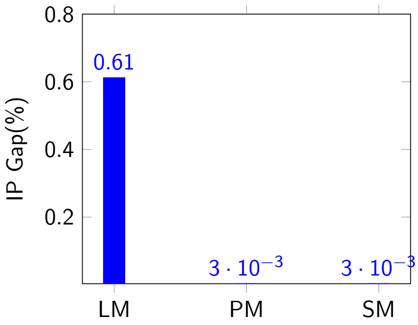
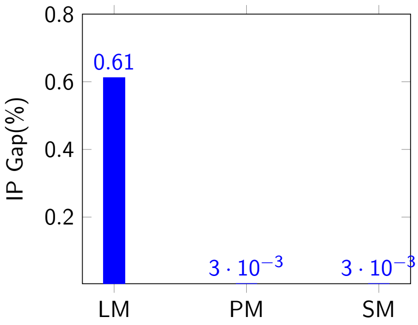


Figure: Average IP Gaps.

- AF: No lower bounds obtained

Results: Large Instances



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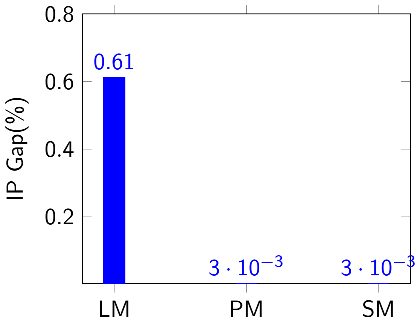

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  - quickly find a lower bound

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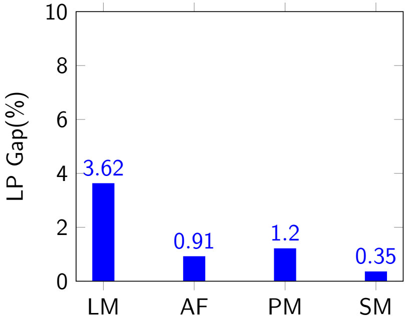

Figure: Average LP gaps.

- AF: decent upper bound

Results: Large Instances

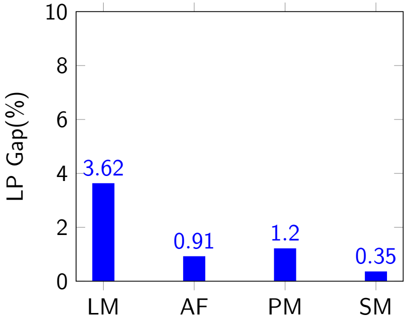

Figure: Average LP gaps.

- AF: decent upper bound
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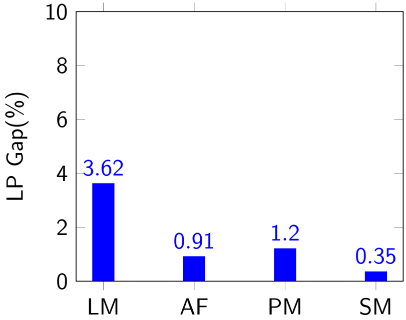
Results: Large Instances



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- AF: decent upper bound
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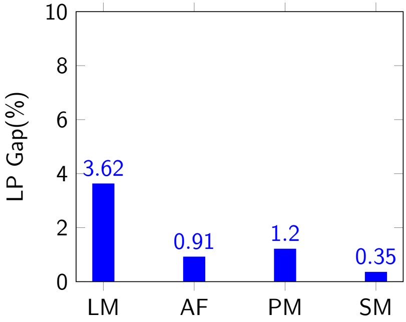

Figure: Average LP gaps.

- AF: decent upper bound
  - but requires a lot of time
- SM: very tight upper bound
- Similar to the case of small instances

Conclusion

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- z^{AF} and z^{LM}?
- Polynomial-size with a decent upper bounds?

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The End

Thank you for listening.

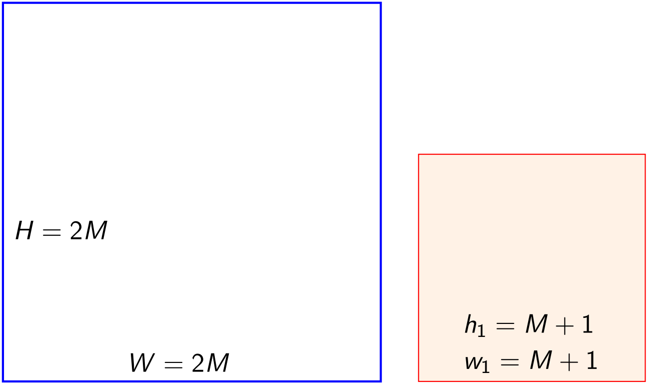
Small Instances

Name	n	W	H	w_{min}	w_{max}	h_{min}	h_{max}	d_{min}	d_{max}	OPT
2	10	40	70	9	31	7	35	1	3	2,535
2s	10	40	70	9	31	7	35	1	3	2,430
3	20	40	70	9	33	11	43	1	4	1,720
3s	20	40	70	9	33	11	43	1	4	2,599
A1s	20	50	60	9	33	11	43	1	4	2,950
A2s	20	60	60	12	33	14	42	1	4	3,423
A3	20	70	80	15	35	14	43	1	4	5,380
A4	20	90	70	9	33	11	43	1	3	5,885
A5	20	132	100	13	69	12	63	1	5	12,553
CHL1	30	132	100	13	69	12	63	1	5	8,360
CHL1s	30	132	100	13	69	12	63	1	5	13,036
CHL2	10	62	55	11	31	9	31	1	3	2,235
CHL2s	10	62	55	11	31	9	31	1	3	3,162
CHL5	10	20	20	1	20	2	14	1	3	363
CHL6	30	130	130	18	69	12	63	1	5	16,572
CHL7	35	130	130	19	57	18	54	1	5	16,728

Large Instances

Name	n	W	H	w_{min}	w_{max}	h_{min}	h_{max}	d_{min}	d_{max}	OPT
ATP30	38	927	152	57	360	7	58	1	9	140,168
ATP31	51	856	964	44	331	50	380	1	9	820,260
ATP32	56	307	124	16	120	6	46	1	9	37,880
ATP33	44	241	983	15	90	52	390	1	9	235,580
ATP34	27	795	456	46	308	22	173	1	9	356,159
ATP35	29	960	649	50	363	34	248	1	9	614,429
ATP36	28	537	244	30	209	20	91	1	9	129,262
ATP37	43	440	881	23	175	51	350	1	9	384,478
ATP38	40	731	358	41	289	19	140	1	9	259,070
ATP39	33	538	501	28	214	48	192	1	9	266,135
ATP40	56	683	138	34	270	6	54	1	9	63,945
ATP41	36	837	367	43	326	32	144	1	9	202,305
ATP42	59	167	291	8	65	21	114	1	9	32,589
ATP43	49	362	917	19	143	46	362	1	9	208,998
ATP44	39	223	496	11	88	29	193	1	9	70,940
ATP45	33	188	578	9	74	49	228	1	9	74,205
ATP46	42	416	514	23	157	40	204	1	9	146,402
ATP47	43	393	554	25	156	32	215	1	9	144,317
ATP48	34	931	254	47	355	18	99	1	9	165,428
ATP49	25	759	449	42	301	23	157	1	9	206,965

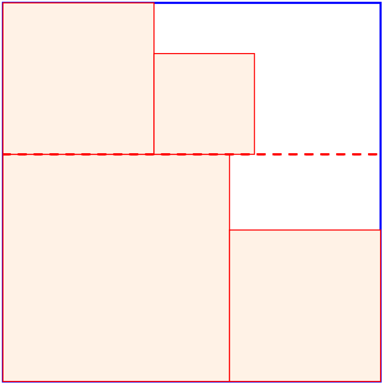
Tight Example



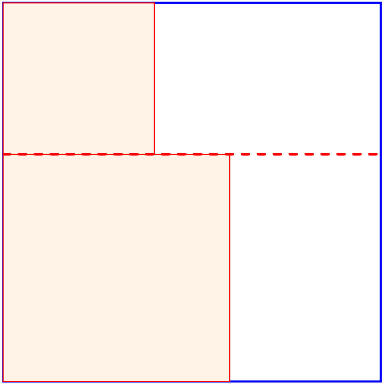
Optimal Objective value: 1

$d_1 = 4$
$p_1 = 1$

Tight Example



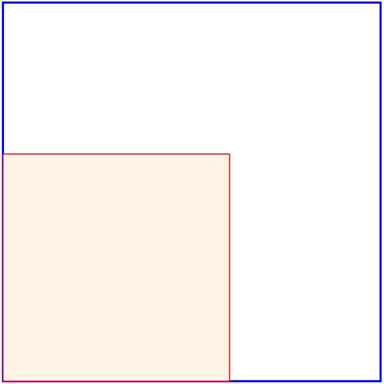
LM $\rightarrow$ 4



PM $\rightarrow$ 2

Feasible Width Pattern: (1)

Tight Example



SM $\rightarrow$ 1

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